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A convergence question inspired by Stieltjes and by value sets in continued fraction theory[1]

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Abstract

Let V be a subset of the complex plane $\mathbb{C}$. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of self-mappings of V; i.e., $f_n(V) \subseteq V$. The question is then: Under what conditions will the sequence

$$F_n(w) := f_1 \circ f_2 \circ \cdots \circ f_n(w); \quad n = 1, 2, 3, \ldots$$

of composite maps converge to a constant function in V? In this paper we give a survey of some of the answers and open problems connected with this question. Such answers have applications in dynamical systems, Schur analysis, continued fractions and other similar structures like infinite exponentials, infinite radicals.

Keywords: Convergence; Continued fractions; Value sets; Iterations; Compositions; Contractive mappings; Linear fractional transformations

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1. Introduction

Let us first look at continued fractions

$$K(a_n/b_n) = \frac{a_1}{b_1} \, {}_{+} \, \frac{a_2}{b_2} \, {}_{+} \, \frac{a_3}{b_3} \, {}_{+} \cdots = \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cfrac{a_4}{b_4 + \ddots}}}}; \quad a_n \neq 0, \tag{1.1}$$

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with approximants

$$\frac{P_n}{Q_n} := \frac{a_1}{b_1} \, \frac{a_2}{+ b_2} + \cdots \frac{a_n}{+ b_n} \quad \text{for } n = 1,2,3,\ldots\,. \tag{1.2}$$

A sequence $\{V_n\}_{n=0}^{\infty}$ of non-empty sets from $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ is called a sequence of *value sets* for $K(a_n/b_n)$ if

$$s_n(V_n) := \frac{a_n}{b_n + V_n} \subseteq V_{n-1} \quad \text{for } n = 1,2,3,\ldots\,. \tag{1.3}$$

If all $V_n = V$, we say that V is a *simple value set* for $K(a_n/b_n)$. If all $V_{2n} = V_0$ and $V_{2n+1} = V_1$, we say that (V_0, V_1) are *twin value sets* for $K(a_n/b_n)$. Since $P_n/Q_n = S_n(0)$, where $S_n(w)$ are linear fractional transformations obtained by compositions

$$S_n(w) := s_1 \circ s_2 \circ \cdots \circ s_n(w) \quad \text{for } n = 1,2,3,\ldots, \tag{1.4}$$

we find that $P_n/Q_n \in V_0$ if $a_n/b_n \in V_{n-1}$. This gives a better chance to determine the asymptotic behavior of $\{P_n/Q_n\}$.

This fruitful concept was introduced by Scott and Wall [43], and skillfully exploited by a number of people, in particular Thron [17, 18, 23–25, 48–53]. It turns out that one often gets more than one asks for. For instance, one may obtain that $\lim_{n\to\infty} S_n(w_n) = \lim_{n\to\infty} S_n(0) =: f$ for every sequence $\{w_n\}$, where w_n belongs to the interior of V_n for each n.

The topic of the present paper is inspired by this value set technique: Let $\{V_n\}$ be a sequence of non-empty subsets of $\hat{\mathbb{C}}$, and let $\{f_n(w)\}_{n=1}^{\infty}$ be a sequence of functions such that $f_n(V_n) \subseteq V_{n-1}$ for all n. Under what conditions will the sequence

$$F_n(w_n) := f_1 \circ f_2 \circ \cdots \circ f_n(w_n); \quad w_n \in V_n, \text{ for } n = 1,2,3,\ldots \tag{1.5}$$

converge to a constant independent of the actual choice of $\{w_n\}$? Let us first consider the case where the sets V_n are open and simply-connected, with $V_n \setminus \{\infty\} \neq \mathbb{C}$, and the functions $f_n(w)$ are holomorphic in V_n. ($f_n(w)$ is holomorphic at $w = \infty$ if $f_n(1/w)$ is holomorphic at $w = 0$.)

In this case, the Riemann mapping theorem ensures the existence of univalent, holomorphic functions φ_n such that $\varphi_n(V_n) = U$, the open unit disk. The compositions $g_n := \varphi_{n-1} \circ f_n \circ \varphi_n^{-1}$ are holomorphic in U with $g_n(U) \subseteq U$. Since $G_n := g_1 \circ g_2 \circ \cdots \circ g_n = \varphi_0 \circ F_n \circ \varphi_n^{-1}$, one can assume that all $V_n = U$, and ask: Under what conditions will $F_n(w)$ converge locally uniformly in U to a constant function?

In this form, the question is related to the following classical result:

The Denjoy–Wolff Theorem (Denjoy [6], Wolff [57]). *Let $f(w)$ be holomorphic in the open unit disk U with $f(U) \subseteq U$. Then f is either a linear fractional transformation mapping U onto U, or f has a fixed point α in the closure $\bar{U}$ of U such that*

$$\lim_{n\to\infty} F_n(w) := \lim_{n\to\infty} f \circ f \circ \cdots \circ f(w) = \alpha \quad \textit{loc. unif. in } U. \tag{1.6}$$

The Denjoy–Wolff Theorem describes the situation where $F_n(w)$ are iterations (1.6) of a single function f. The more general question (1.5) was asked by Thron in [53], although his interest was mainly with continued fractions, infinite exponentials [50] and infinite radicals [41]. Later it was taken up in [1, 2, 7–10, 27], independently. (My paper was first submitted in January 1988, but was rejected on the grounds that this question had no interest.)

The purpose of the present paper is to present some of the answers known so far. We start with Stieltjes' beautiful convergence result for continued fractions in Section 2. In Section 3, we describe some natural extensions of Stieltjes' result, the parabola theorems for continued fractions $K(a_n/1)$. We also show how some other classical results can be proved by means of value sets and Stieltjes' ideas. Section 4 is devoted to the general problem of convergence of $\{F_n(w)\}$. It turns out that we still can get positive results, even if $\{f_n\}$ is not holomorphic. Section 5 describes some results obtained for the special case where all f_n are linear fractional transformations. This is an important case, both for applications and to gain experience about what to except. In Section 6, we return to the question of convergence of continued fractions $K(a_n/1)$. Finally, in Section 7 we say a few words about how to find/choose value sets.

2. Stieltjes' Convergence Theorem for continued fractions

Thomas Jan Stieltjes considered continued fractions of the form

$$\frac{1}{b_1 z} \,\underset{+}{}\, \frac{1}{b_2} \,\underset{+}{}\, \frac{1}{b_3 z} \,\underset{+}{}\, \frac{1}{b_4} \,\underset{+}{}\, \cdots; \quad b_n > 0 \tag{2.1}$$

in his very important paper [47]. An equivalent form is $K(a_n\zeta/1)$, where $\zeta = 1/z$, $a_1 = 1/b_1$ and $a_n = 1/b_{n-1}b_n$ for $n \geqslant 2$. Continued fractions of these forms are now called *Stieltjes fractions*, or just *S-fractions* for short, in honor of this great mathematician. Stieltjes encountered the continued fraction (2.1) when he studied his moment problem. It was therefore important to establish criteria for the convergence or divergence of (2.1). Stieltjes proved:

Stieltjes' Convergence Theorem for continued fractions.

(A) *The sequences* $\{P_{2n}(z)/Q_{2n}(z)\}$ *and* $\{P_{2n+1}(z)/Q_{2n+1}(z)\}$ *of even and odd approximants for* (2.1) *converge locally uniformly in* $S := \{z \in \mathbb{C}\colon |\arg z| < \pi\}$ *to holomorphic functions.*

(B) *The two limits in Part* A *coincide, and thus the continued fraction* (2.1) *converges at a point* $z \in S$, *if and only if* $\sum b_n = \infty$.

(C) *If* (2.1) *converges at a point* $z \in S$, *then it converges locally uniformly in* S *to a holomorphic function* $f(z)$.

(Part C is of course a trivial consequence of Part A and B.) Not only is this an extremely beautiful result, Stieltjes also gave integral representations for the limit functions and his proof was indeed impressive. The main steps in his proof are:

(1) Convergence for $z = 1$.

This question was settled by Stern [46] already in 1848, but Stieltjes included the proof for completeness. It is based on the following expression for the approximants P_n/Q_n (see for example [36, p. 41]):

$$\frac{P_n}{Q_n} = \frac{1}{Q_0 Q_1} - \frac{1}{Q_1 Q_2} + \frac{1}{Q_2 Q_3} - \cdots + \frac{(-1)^{n-1}}{Q_{n-1} Q_n}. \tag{2.2}$$

According to the Seidel–Stern Theorem [36, Satz 2.12, p. 48], this is an alternating series with monotonely decreasing terms when $z = 1$. Hence, $\{P_{2n}/Q_{2n}\}$ and $\{P_{2n+1}/Q_{2n+1}\}$ converge to finite limits. They have a common limit iff $Q_{n-1}Q_n \to \infty$ as $n \to \infty$, which is equivalent to $\sum b_n = \infty$.

(2) Convergence for $z = x > 0$.

This case follows from Part 1 by the substitution $b_{2n-1}x = \tilde{b}_{2n-1}$.

(3) Uniform convergence for $z = x + iy$ with $x \geqslant \lambda > 0$.

Stieltjes settled this case by a clever comparison to the case $z = \lambda$.

(4) Locally uniform convergence for $z \in S$.

This is an ingeneous step! Stieltjes proves his famous theorem for the extension of the domain of convergence:

Stieltjes' Convergence Extension Theorem. *The series $\sum_{k=1}^{\infty} f_k(z)$ converges locally uniformly to a holomorphic function $F(z)$ in some open, connected set S if the following three conditions hold:*

(i) *each $f_k(z)$ is holomorphic in S,*

(ii) *$\sum f_k(z)$ converges uniformly in a non-empty disk $|z - z_0| \leqslant R$ contained in S, and*

(iii) *for each compact subset S' of S there exists a uniform bound $L_{S'}$ such that $|\sum_{k=1}^{n} f_k(z)| \leqslant L_{S'}$ for all $n \in \mathbb{N}$ and all $z \in S'$.*

Stieltjes managed to confirm the locally uniform boundedness for $\{P_n(z)/Q_n(z)\}$ in condition (iii). The locally uniform convergence was thus fully established.

3. The value set idea for continued fractions

Stieltjes' Convergence Extension Theorem was improved by Vitali to what is now often referred to as:

The Stieltjes–Vitali Theorem. *The sequence $\{F_n(z)\}$ converges locally uniformly to a holomorphic function in some open, connected set $S \subseteq \mathbb{C}$ if the following three conditions hold:*

(i) *each $F_n(z)$ is holomorphic in S,*

(ii) *$\{F_n(z)\}$ converges to a finite value for every $z \in K$, where $K \subseteq S$ is an infinite set with at least one limit point in S, and*

(iii) *there exist two distinct points $a, b \in \mathbb{C}$ such that $F_n(z) \neq a$, $F_n(z) \neq b$ for all n and all $z \in S$.*

In both forms, these results are dynamite in combination with the concept of value sets for continued fractions. Let $K(a_n/b_n)$ be a continued fraction with corresponding value sets $\{V_n\}$. If a_n and b_n are functions of some variable z, then also $\{V_n\}$ may depend on z for z in some open, connected set S. If for every compact subset $S' \subset S$, $V = V_0(z)$ omits two values $a_{S'}$, $b_{S'}$ in

$\mathbb{C}$, then condition (iii) in the Stieltjes–Vitali Theorem is settled for $\{S_n(w_n)\}$; $w_n \in V_n$. (The fact that V_n may vary with n can again be handled by use of the Riemann Mapping Theorem, as in the introductory section.)

Since $F_n(z) = S_n(z)$ are linear fractional transformations in this case, we also allow $\infty \in V_n$. If $\infty \in V_0$, and V_0 omits three points in $\mathbb{C}$, then a rotation φ of the Riemann sphere gives $\infty \notin \varphi(V_0)$, and we can apply the Stieltjes–Vitali Theorem to $F_n := \varphi \circ S_n \circ \varphi^{-1}$.

Example 1. Henrici and Pfluger [11] proved that the angular opening

$$V(z) := \begin{cases} \{w \in \mathbb{C}:\ 0 \leqslant \arg w \leqslant \arg z\} & \text{if } 0 \leqslant \arg z \leqslant \pi \\ \{w \in \mathbb{C}:\ \arg z \leqslant \arg w \leqslant 0\} & \text{if } -\pi \leqslant \arg z \leqslant 0 \end{cases}$$

is a simple value set for the S-fraction $K(a_n z/1)$ (all $a_n > 0$) for a fixed $z \in S := \{z \in \mathbb{C}: |\arg z| < \pi\}$. Hence

$$S_n(z, w) := \frac{a_1 z}{1} + \frac{a_2 z}{1} + \cdots + \frac{a_n z}{1 + w} \in V(z); \quad n = 1, 2, 3, \ldots$$

if $w \in V(z)$. This means that $S_n(z, w) \in S$ for $w \geqslant 0$ for all $z \in S$. This observation implies that condition (iii) in the Stieltjes–Vitali Theorem holds for $z \in S$. Condition (ii) is satisfied by step 2 in Stieltjes' proof if and only if $\sum b_n = \infty$, where b_n is given by (3.2) below. Hence, Stieltjes' Convergence Theorem follows immediately.

As we see, these are powerful tools indeed! It was Leighton and Thron [24] who suggested to use them in combination. Their first example belongs to the class of parabola theorems for continued fractions $K(a_n/1)$. Let us take the following member of this class as our starting point.

The Simple Parabola Theorem. *Let $\alpha \in (-\frac{1}{2}\pi, \frac{1}{2}\pi)$ be fixed, and let $K(a_n/1)$ be a continued fraction from*

$$P_\alpha := \{w \in \mathbb{C}: |w| - \Re(w e^{-i2\alpha}) \leqslant \tfrac{1}{2}\cos^2\alpha\}. \tag{3.1}$$

(That is, $0 \neq a_n \in P_\alpha$ for all n.) Then:

(A) *The sequences $\{P_{2n}/Q_{2n}\}$ and $\{P_{2n+1}/Q_{2n+1}\}$ of even and odd approximants for $K(a_n/1)$ converge to finite limits.*

(B) *The two limits in Part* A *coincide, and thus $K(a_n/1)$ converges, if and only if*

$$\sum |b_n| = \infty \quad \text{where } b_n := \prod_{k=1}^{n} a_k^{(-1)^{n+k+1}}. \tag{3.2}$$

P_α is bounded by a parabola passing through $z = -\frac{1}{4}$, with focus at the origin, vertex at $-\frac{1}{4}e^{i2\alpha}\cos^2\alpha$ and axis along the ray $\arg w = 2\alpha$ (Fig. 1). Hence the name of the theorem, which clearly generalizes the pointwise convergence aspect of Stieltjes' Convergence Theorem. Its proof is based on the fact that

$$V_\alpha = \{w \in \mathbb{C}: \Re(w e^{-i\alpha}) \geqslant -\tfrac{1}{2}\cos\alpha\} \tag{3.3}$$

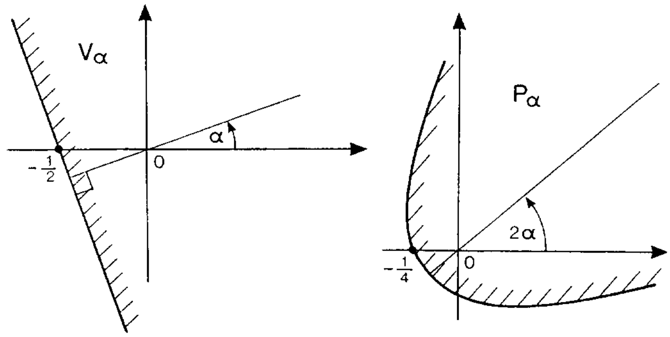

Fig. 1.

is a simple value set for $K(a_n/1)$. The quantities b_n in (3.2) are the coefficients of the equivalent continued fraction $K(1/b_n)$; i.e., $a_1 = 1/b_1$ and $a_n = 1/b_{n-1}b_n$ for $n \geqslant 2$.

This theorem is very useful to prove convergence of continued fractions. The sets P_α are actually best possible, in the sense that they cannot be enlarged, not even by including just one extra point and still guarantee convergence if (3.2) holds [28].

The very first parabola theorem was due to Scott and Wall [42]. It is identical to the Simple Parabola Theorem for the special case $\alpha = 0$. It was proved by a technique which amounts to applying the series (2.2).

Leighton and Thron [24] pointed out that the Stieltjes–Vitali Theorem gives a possibility to derive convergence results also for the rotated parabolas, and that one in fact obtains uniform convergence for $K(a_n(z)/1)$ when all $a_n(z)$ lie in a compact subset of the interior of P_α for z in some set S.

Paydon and Wall [34], and independently Leighton and Thron [25] thus came up with the Simple Parabola Theorem, but with P_α replaced by a compact subset of its interior. They actually proved:

The Cardioid Theorem (Paydon and Wall [34], Leighton and Thron [25]). *Let $K(a_n z/1)$ be a continued fraction with all elements a_n in P_0 given by* (3.1). *Then $K(a_n z/1)$ converges locally uniformly with respect to z in the cardioid region* (*see Fig.* 2)

$$S := \{z = re^{i\theta}: r < \tfrac{1}{2}(1 + \cos\theta)\}; \quad -\pi < \theta < \pi \tag{3.4}$$

to a holomorphic function in S, provided (3.2) *holds. If the series in* (3.2) *converges, then the sequences of even and odd approximants for $K(a_n z/1)$ still converge locally uniformly with respect to $z \in S$ to holomorphic functions in S.*

Their version of the Simple Parabola Theorem was a corollary of this result.

The Cardioid Theorem is important in its own right. A stronger version can easily be obtained from the Simple Parabola Theorem. If all $a_n \in P_\alpha$ for another value of α, then we get

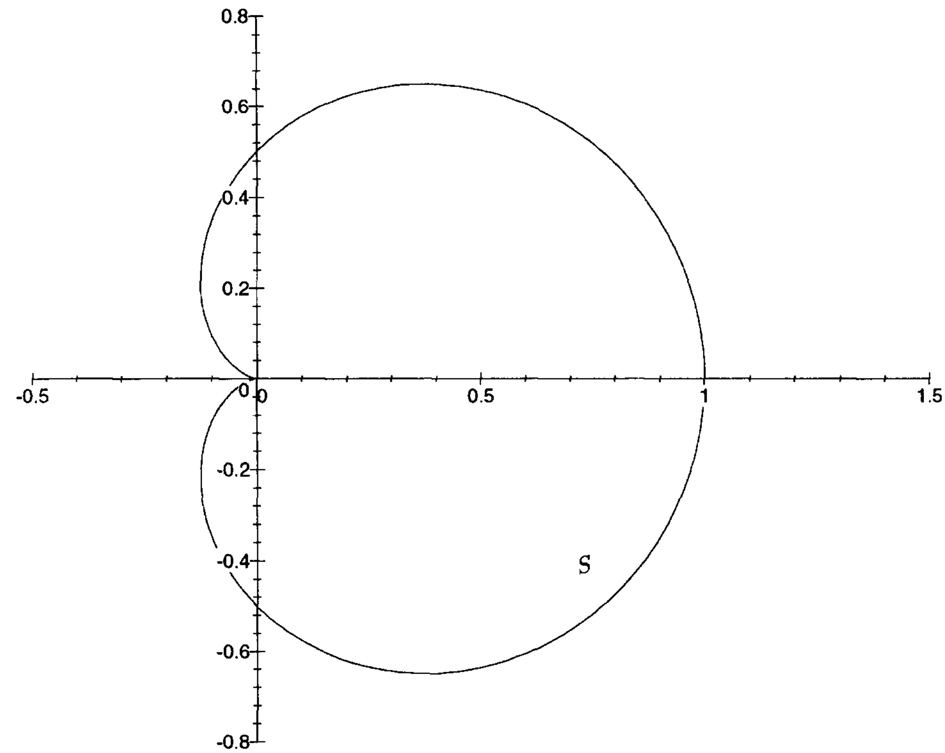

Fig. 2.

locally uniform convergence in

$$S_\alpha := \left\{ z = re^{i\theta} \colon r < \frac{\cos^2(\alpha + \theta/2)}{\cos^2\alpha} \right\}; \quad -\pi - 2\alpha < \theta < \pi - 2\alpha. \tag{3.4'}$$

Similarly, if all a_n are contained in a slimmer parabolic region, then S_α can be made correspondingly fatter.

Not much later, Thron proved the Simple Parabola Theorem in its full generality [49]. This beautiful result was a corollary to a twin convergence result, which he proved by a clever use of the Stieltjes–Vitali Theorem.

No question, the value set technique in combination with the Stieltjes–Vitali Theorem works miracles. On the other hand, having obtained this much, we want more! For instance, does $K(a_n z/1)$ converge uniformly in the closure of S in the Cardioid Theorem? What is the speed of convergence? For this we can no longer rely on the Stieltjes–Vitali Theorem. But the value sets also open up for another approach. From (1.3) we find that the sets $S_n(V_n)$ are nested in the sense that $S_n(V_n) = S_{n-1} \circ s_n(V_n) \subseteq S_{n-1}(V_{n-1})$. Hence the limit $\Delta := \lim_{n\to\infty} S_n(\bar{V}_n)$ exists and is non-empty. ($\bar{A}$ denotes the closure of a set A in $\mathbb{C}$.)

If Δ is a one-point set, the limit-point case, then $\{S_n(w_n)\}$ converges, uniformly with respect to $w_n \in \bar{V}_n$, to this point f in Δ. Moreover, $|f - S_n(w_n)| \leqslant \operatorname{diam} S_n(\bar{V}_n)$. In particular, $\{S_n(0)\}$ converges to f if $a_n/b_n \in \bar{V}_{n-1}$ for all n.

If Δ contains more than one point, the situation is more complicated, $\{S_n(w_n)\}$ may still converge, but one has to investigate further.

It took 15 years, and clever manipulation before the following result appeared:

The Uniform Parabola Sequence Theorem (Thron [51]). *Let $\alpha \in (-\frac{1}{2}\pi, \frac{1}{2}\pi)$, $g_0 > 0$ and $0 < g_n < 1$ for all $n \in \mathbb{N}$ be given, and let*

$$V_{\alpha,n} := \{w \in \mathbb{C}\colon \Re(we^{-i\alpha}) \geqslant -g_n \cos\alpha\} \quad \textit{for } n = 0, 1, 2, \ldots . \tag{3.5}$$

Let further $K(a_n/1)$ be a continued fraction with

$$a_n \in P_{\alpha,n} := \{w \in \mathbb{C}\colon |w| - \Re(we^{-i2\alpha}) \leqslant 2g_{n-1}(1-g_n)\cos^2\alpha\} \tag{3.6}$$

for all n. Then $\{V_{\alpha,n}\}$ are value sets for $K(a_n/1)$, and the diameter of $S_n(V_{\alpha,n})$ is bounded by

$$\operatorname{diam}(S_n(V_{\alpha,n})) \leqslant \frac{|a_1|/((1-g_1)\cos\alpha)}{\prod_{k=2}^{n}(1 + (d_{k-1}g_{k-1}(1-g_k)\cos^2\alpha)/|a_k|)}; \quad d_k := G_k \Big/ \sum_{j=0}^{k-1} G_j, \tag{3.7}$$

where $G_k := \prod_{n=1}^{k}(1-g_n)/g_n$.

If the bound in (3.7) vanishes as $n \to \infty$, we have the limit-point case, and thus $K(a_n/1)$ converges. Then $|f - P_n/Q_n| \leqslant \operatorname{diam}(S_n(V_{\alpha,n}))$. This happens for instance if $\sum G_n = \infty$, $\liminf g_n > 0$, $\limsup g_n < 1$ and $\{a_n\}$ is bounded.

The bound (3.7) can also be used to prove uniform convergence of continued fractions $K(a_n(z)/1)$ with $a_n(z) \in P_{\alpha(z),n}$ for all n under proper conditions. The choice $g_n := \frac{1}{2}$ for all n gives $P_{\alpha,n} = P_\alpha$ in (3.1) for all n. In this case the bound (3.7) implies that

$$\left| f - \frac{P_n}{Q_n} \right| \leqslant \frac{2|a_1|/\cos\alpha}{\prod_{k=2}^{n}(1 + (\cos^2\alpha)/(4(k-1)|a_k|))} \tag{3.8}$$

if $K(a_n/1)$ converges to f. In particular, the Cardioid Theorem extends to the boundary of S; i.e., it gives uniform convergence for all z in the closure of S in (3.4), if the bound in (3.8) approaches 0 as $n \to \infty$.

There is no law saying that the value sets have to be bounded by circular arcs or straight lines, even though such sets are simpler to work with. In his thesis, Córdova Yévenes gave the following impressive version of the Simple Parabola Theorem.

Theorem 3.1 (Córdova Yévenes [5]). *Let V be a closed region (open, connected set) in $\mathbb{C}$ with boundary ∂V such that*

(i) *∂V is a smooth Jordan curve in $\hat{\mathbb{C}}$ with continuous curvature, symmetric with respect to $-\frac{1}{2}$, with $-\frac{1}{2} \in \partial V$ and $\infty \in \partial V$, and*

(ii) *for every $w \in \partial V$, the curve $-w^2/(1 + \partial V)$ has only one point in common with ∂V (which then has to be w), and the curvature of this curve at this point w is larger than the (signed) curvature of ∂V at w.*

Let $E := \{w \in \mathbb{C}\colon w/(1+V) \subseteq V\}$. *Then every continued fraction* $K(a_n/1)$ *from* E *with* $\{a_n\}$ *bounded, converges. Indeed, we get the limit-point case.*

The choice $V = V_\alpha$ given by (3.3) gives back the Simple Parabola Theorem for the special case where $\{a_n\}$ is bounded. Córdova Yévenes' generalization is extremely interesting, and opens up for a whole line of questions! Can the boundedness of $\{a_n\}$ in his theorem be replaced by something more similar to the condition (3.2)? Does $K(a_n(z)/1)$ converge uniformly with respect to z if all $a_n(z)$ are contained in some bounded subset of E? This is clearly true for compact subsets of the interior of E by the Stieltjes–Vitali Theorem. Under what conditions will Theorem 3.1 still hold with V replaced by a sequence $\{V_n\}$ of value sets? For which sets V and E can the curve $-w^2/(1+\partial V)$ have more than one point of contact with ∂V for a $w \in \partial V$? Which sets V give nice, large element sets E? It seems that continuity of the curvature is asking too much? I sincerely hope that Córdova Yévenes will look closer at some of these questions. In the meantime we can enjoy the following result by Lange:

Theorem 3.2 (Lange [21]). *The conclusions* (A) *and* (B) *in the Simple Parabola Theorem also hold for the continued fraction* $K(a_n/1)$ *if all* $a_n = -c_n^2$, *where* $c_n \in C^+(d,0) \cup C^-(d,\varepsilon)$ *for an* $\varepsilon > 0$, *where* $C^\pm(d,\mu) := \{c = x + \mathrm{i}y\colon 0 \leqslant x \leqslant g_\mu(y),\ y \in \mathbb{R}^\pm\}$, *and the curve* $x = g_\mu(y)$ *has the parametrization*

$$x(t) = t + \mu f'(t)/\sqrt{1+(f'(t))^2}, \qquad y(t) = f(t) - \mu/\sqrt{1+(f'(t))^2} \tag{3.9}$$

for $|t - \frac{1}{2}| < d$ *with* $f(t) = \tan(\pi(\frac{1}{2} - \frac{1}{2}t)/d)$ *for a* $\mu \geqslant 0$ *and* $0 < d \leqslant \frac{1}{2}$.

Lange proved that $V(d)$ given by

$$1 + V(d) := \{w = x + \mathrm{i}y\colon x > g_0(y),\ y \in \mathbb{R}\} \tag{3.10}$$

is a value set for $K(a_n/1)$. His theorem is thus a consequence of the Stieltjes–Vitali Theorem. According to Theorem 3.1 we can set $\varepsilon := 0$ in Theorem 3.2 if $\{a_n\}$ is bounded. That is, $K(-c_n^2/1)$ converges if $\{c_n\}$ is a bounded sequence from $C^+(d,0) \cup C^-(d,0)$ (see Fig. 3).

Lange [21] also proved a second result of similar nature. This time $K(a_n/1)$ had elements $a_n = -c_n^2$, where c_n belonged to what he called a Worpitzky strip. We refer to his paper for further details.

The value set technique also gives new proofs for a number of classical convergence criteria:

(1) The Worpitzky Theorem [58]: The disk $E := \{w \in \mathbb{C}\colon |w| \leqslant \frac{1}{4}\}$ is a simple, uniform convergence set for continued fractions $K(a_n/1)$. Value set: $V := \{w \in \mathbb{C}\colon |w| \leqslant \frac{1}{2}\}$.

(2) The van Vleck Theorem [54]: $K(1/b_n)$ converges if all $b_n \in V_\varepsilon := \{w \in \mathbb{C}\colon |\arg w| \leqslant \frac{1}{2}\pi - \varepsilon\}$ for an $\varepsilon > 0$, and $\sum |b_n| = \infty$. Value set: $V = V_\varepsilon$.

(3) The Śleszyński–Pringsheim Criterion [38], [45]: $K(a_n/b_n)$ converges if $|a_n| + 1 \leqslant |b_n|$ for all n. Value set: $V = \bar{U}$, the closed unit disk.

(4) The Leighton–Wall twin convergence regions [26]: $K(a_n/1)$ converges to a finite limit if $|a_{2n-1}| \leqslant \frac{1}{4}$ and $|a_{2n}| \geqslant 25/4$ for all n. Twin value sets: $V_0 := \{w \in \mathbb{C}\colon |w| \leqslant \frac{1}{2}(5 - \sqrt{24})\}$, $V_1 := \{w \in \mathbb{C}\colon |w + 1| \geqslant \frac{1}{2}(5 + \sqrt{24})\}$.

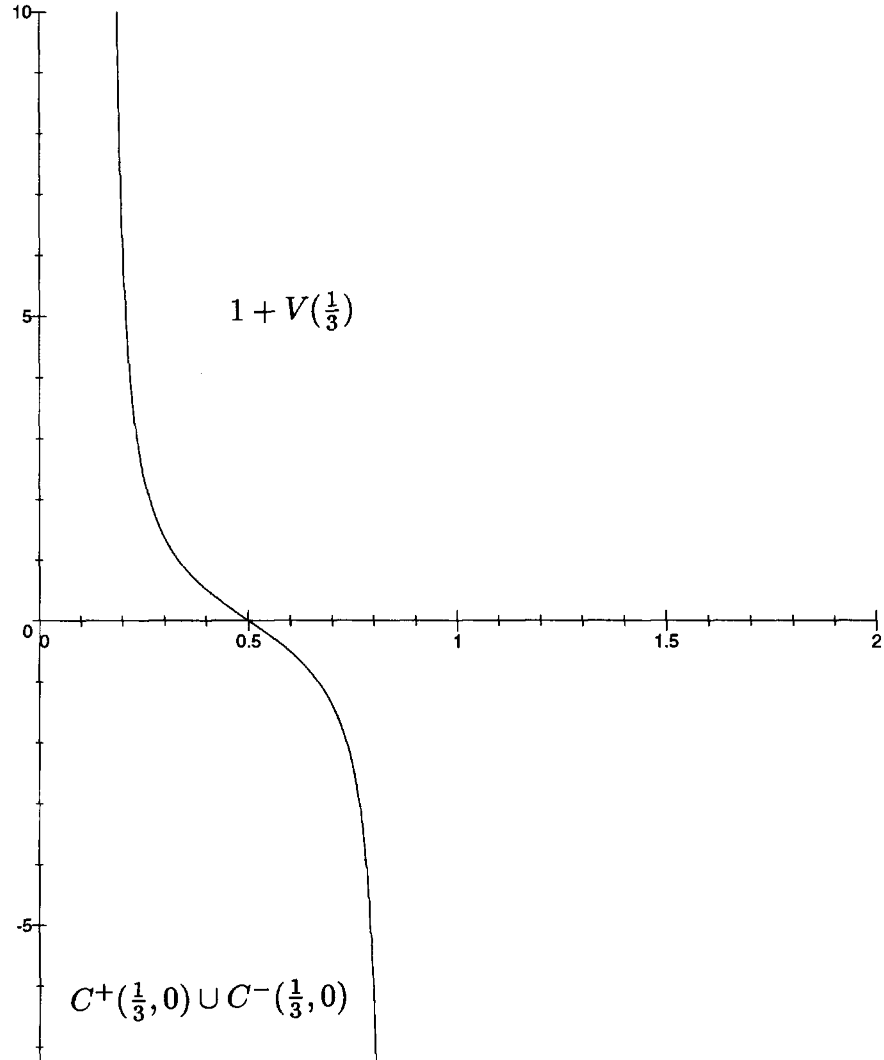

Fig. 3.

(5) The Twin Limaçon Theorem, initiated by Thron [48, 49], generalized further in [23, 44] and finally proved in its present form by Lange [20]: The sets $E_1 := \{w \in \mathbb{C}\colon\ w = c^2,\ |c \pm ia| \leqslant \rho\}$ and $E_2 := \{w \in \mathbb{C}\colon\ w = c^2,\ |c \pm i(1+a)| \geqslant \rho\}$ are uniform twin convergence sets if $a \in \mathbb{C}$ and $\rho > 0$ satisfies $|a| < \rho < |1+a|$. Twin value sets: $V_0 = \{w \in \mathbb{C}\colon\ |w-a| \leqslant \rho\}$, $V_1 = \{w \in \mathbb{C}\colon\ |w+1+a| \geqslant \rho\}$.

The traditional way to define convergence is to require that $\{S_n(0)\}$ converges. More generally, one can ask for convergence of $\{S_n(w)\}$ or even $\{S_n(w_n)\}$, as long as one stays away from certain dangerous sequences $\{w_n\}$. (See [15, 34].) Actually, the following holds:

Theorem 3.3 (Jacobsen [16]). *Let $K(a_n/1)$ be a continued fraction and V be a corresponding value set. If V is bounded, and $\{S_n(w)\}$ converges for a w in the interior of V, then $\{S_n(0)\}$ also converges to the same value.*

The same conclusion holds if V is bounded and $\lim S_n(u) = \lim S_n(v)$ for two distinct points u, $v \in V$ [29]. Hence, the condition $w_n \in V$ is not necessary to apply the value set technique. One immediately gets results like the following (see [16] for further details):

Theorem 3.4. *Let $\emptyset \neq V \subseteq \mathbb{C}$ be bounded, and let G be a compact subset of the interior of $E := \{a \in \mathbb{C}: a/(1+V) \subseteq V\}$. Then every continued fraction $K(a_n/1)$ from G converges.*

(Of course, if E or its interior is empty, then there are no continued fractions from G.)

Theorem 3.5. *Let $\emptyset \neq V_0 \subseteq \mathbb{C}$ be bounded, and $V_1 \subseteq \mathbb{C}$. Let G_1 and G_2 be compact subsets of the interior of $E_1 := \{a \in \mathbb{C}: a/(1+V_1) \subseteq V_0\}$ and $E_2 := \{a \in \mathbb{C}: a/(1+V_0) \subseteq V_1\}$, respectively. Then every continued fraction $K(a_n/1)$ from (G_1, G_2) (i.e., all $a_{2n-1} \in G_1$, $a_{2n} \in G_2$) converges.*

The proofs are simple and typical. We look briefly at the proof of Theorem 3.4. Let $K(a_n/1)$ be a continued fraction from G. It is easy to find a circular disk $D \subseteq E$ such that all continued fractions from D converges. Let $a_n(z)$ be holomorphic functions such that $a_n(z) \in E$ for $|z| \leqslant 1 + \varepsilon$, $a_n(1) = a_n$ and $a_n(z) \in D$ for $|z| < \varepsilon$ for an $\varepsilon > 0$. Then it follows by the Stieltjes–Vitali Theorem that $\{S_n(w)\}$ converges to a constant function in V. The convergence of $\{S_n(0)\}$ is then a consequence of Theorem 3.3.

The first result in this direction dates from 1982 [14]. A simpler (and better) result is presented in [17]. It is subsumed in the following result:

Theorem 3.6 (Lorentzen and Ruscheweyh [32]). *Let $V := \{w \in \mathbb{C}: |w - \Gamma| < R\}$, where $\Re(\Gamma) > -\frac{1}{2}$ and $0 < R < |1 + \Gamma|$. Then every continued fraction $K(a_n/1)$ with $a_n/(1+V) \subseteq V$ for all n, converges.*

(The conditions on Γ and R are just needed to obtain that there exist non-periodic continued fractions $K(a_n/1)$ with the required mapping property.)

Lange [22] has later shown that $E := \{a \in \mathbb{C}: a/(1+V) \subseteq V\}$ is a subset of P_α in (3.1) with $\alpha = \arg(\Gamma + \frac{1}{2})$. Hence E is actually a simple, uniform convergence set.

4. More general functions

Let us now turn to the case where $\{f_n\}$ is a sequence of self-mappings of some region (open, connected set) V. Let us first assume that all f_n are analytic in the open unit disk $V = U$. The following result is then no surprise:

Theorem 4.1 (Baker and Rippon [1], Lorentzen [27]). *Let $\{f_n\}$ be a sequence of functions analytic in U such that all $f_n(U) \subseteq K \subset U$, where K is compact. Then $\{F_n(w)\}$ given by* (1.5) *converges uniformly in U to a constant function.*

One also has a bound for the speed of convergence,

$$|F_n(z) - \lim F_n(z)| \leqslant 2r^n \quad \text{where } r := \max\{|w|: w \in K\}. \tag{4.1}$$

Baker and Rippon also proved the more general result:

Theorem 4.2 (Baker and Rippon [1]). *Let $\{f_n\}$ be a sequence of analytic self-mappings of U such that no sequence of the form*

$$\mu_k := f_{1,k} \circ \cdots \circ f_{n(k),k}; \quad f_{j,k} \in \{f_n: n = 1,2,3, \dots\}; \; k = 1,2,3,\dots$$

converges in U to a constant limit $\alpha \in \partial U$ or to the identity function. Then $F_n(w)$ converges locally uniformly in U to a constant function.

My favorite result in their interesting paper is however the following, which is much simpler to apply. It is related to Theorem 4.1, but K is replaced by $f_n(U)$ which may approach the boundary of U at a controlled rate, at the cost that all f_n be univalent. It uses the notation $\hat{A}$ for the complement of the unbounded component of $\mathbb{C} \backslash \bar{A}$ for a bounded set $A \subset \mathbb{C}$.

Theorem 4.3 (Baker and Rippon [1]). *Let $\{f_n\}$ be a sequence of analytic, univalent self-mappings of U, such that the modulus M_n of the ring domain $U \backslash \widehat{f_n(U)}$ satisfies $\sum M_n = \infty$. Then $F_n(w)$ converges uniformly in U to a constant function.*

Of course, if $\widehat{f_n(U)} \cap \partial U \neq \emptyset$, then $M_n := 0$. A year later, the same authors proved

Theorem 4.4 (Baker and Rippon [2]). *Let $\{f_n\}$ be a sequence of analytic self-mappings of a convex region V such that*

(i) *each f_n has a continuous extension to $\bar{V}$ with $|f_n'(w)| \leqslant 1$ in V, and*
(ii) *there exists a point $w_0 \in V$ such that $\sup_{n \geqslant 1} |f_n'(w_0)| < 1$.*

Then $F_n(w)$ converges uniformly in $\bar{V}$ to a constant function.
If V, in particular, is a circular disk, then

$$\operatorname{diam}(F_n(V)) = \mathrm{O}(n^{-1/2}) \quad \text{as } n \to \infty. \tag{4.2}$$

One must expect that the speed of convergence depends on the shape of the domain V. This is evident from an interesting paper by Beardon [4].

Also these results give rise to a string of new questions. Can the condition $\sum M_n = \infty$ be replaced by some condition involving $R_n := \sup\{|f_n(w)|: w \in U\}$? Can the condition that f_n be univalent be removed? Actually, what is so special with f_n being analytic? At least in the Denjoy–Wolff setting, what is essential is the contractive property of f, a purely geometric property. As pointed out by

Beardon [3], the analyticity of f_n ensures that unless f_n is an isometry in U with respect to the hyperbolic metric ρ, it is a contraction

$$\rho(f_n(z), f_n(w)) < \rho(z, w) \quad \text{for } z \neq w. \tag{4.3}$$

So, let us start with a metric space (X, d), and let $f_n: X \to X$ be contraction maps. Beardon [4] proved the following result for the case where f_n belongs to the family $S(X)$ of maps $f: X \to X$ with the property $\rho(f(x), f(y)) \leqslant \rho(x, y)$ for all $x, y \in X$, where ρ is the metric of uniform convergence on X:

Theorem 4.5 (Beardon [4]). *Let $\mathscr{F}$ be a closed subset of $S(X)$, and let $\{f_n\}$ be a sequence from $\mathscr{F}$. Then either*
(a) *$\{F_n(w)\}$ converges uniformly in (X, d) to a constant function, or*
(b) *$d(f(x), f(y)) = d(x, y)$ for some $f \in \mathscr{F}$ and some $x, y \in X$, $x \neq y$.*

With this theorem we can, for instance, prove convergence results for continued fractions with vector or matrix elements. If we let X be a convex set $\subseteq \mathbb{C}$ and d be the Euclidean metric, then Beardon [4] shows that possibility (b) in Theorem 4.5 is ruled out. So also if $(X, d) = (D, d)$, where D is a convex set in $\mathbb{R}^n$ whose boundary ∂D has no linear segment of positive length, and d is the Euclidean metric in $\mathbb{R}^n$.

However, in all these theorems difficult boundary cases have been excluded. If we do not restrict our class of functions to such an extent, the picture gets considerably more complicated.

Example 2. Let $f_n(w) := a_n(1 - w)/(1 + a_n(1 - w))$ for all n, where $a_n := n(n + 2)$. Then $f_n(U) = \{w \in \mathbb{C}: |w - r_n| < r_n\}$, where $r_n := \frac{1}{2}(1 - 1/(1 + 2a_n))$. In other words, all $f_n(U) \subseteq \mathscr{D} := \{w \in \mathbb{C}: |w - \frac{1}{2}| < \frac{1}{2}\}$. Still, $F_n(w)$ does not converge for any $w \in U$, since $f_n = \varphi \circ s_n \circ \varphi^{-1}$, where $\varphi(w) = w/(1 + w)$ and $s_n(w) = a_n/(1 + w)$, and $S_n(w) := s_1 \circ s_2 \circ \cdots \circ s_n(w)$ diverges by the Simple Parabola Theorem.

On the other hand, if $a_n := n$ for all n, then still $f_n(U) \subseteq \mathscr{D}$, and this time $F_n(w)$ converges locally uniformly in U by the same theorem.

5. Linear fractional transformations

The case where all f_n are linear fractional transformations is particularly interesting. For one thing, it is important in applications, such as continued fractions and Schur analysis. Their mapping properties also make them well suited to study. Furthermore, they include the extreme case in the Denjoy–Wolff Theorem, where $f(w)$ is a linear fractional transformation, mapping the unit disk onto itself. Hence, studies of this case will tell us much about what to expect in more general cases. Finally, we may obtain better results than in the general case of contraction maps in a metric space, since the class of functions is so restrictive.

When we restrict our family of functions in this way, the choice $V = U$ is a severe restriction. Still, it is of interest to see what happens in this case. The first result to be mentioned here is due to Hillam and Thron.

Theorem 5.1 (Hillam and Thron [12]). *Let $\{f_n\}$ be a sequence of linear fractional transformations mapping U into itself, such that $f_n(\infty) = k$ for all n, for some fixed $k \in U$. Then $F_n(w)$ converges locally uniformly in U to a constant function.*

Their condition $f_n(\infty) = k$ is clearly aimed at continued fractions $K(a_n/b_n)$, where $s_n(w) = a_n/(b_n + w)$ satisfies $s_n(\infty) = 0$. It can easily be weakened. This was for instance done in [14]. The following result represents a considerable generalization. I proved it for a conference we had in Loen, only to discover that it is equivalent to a far earlier result due to Jones and Thron:

Theorem 5.2 (Jones and Thron [18], Lorentzen [30]). *Let $\{f_n\}$ be a sequence of linear fractional transformations mapping the unit disk U into itself. If there exists a sequence $\{w_n\}$ of numbers from $\hat{\mathbb{C}}$ such that*

$$\liminf \| w_n | - 1 | > 0 \quad \text{and} \quad \liminf \| f_n(w_n) | - 1 | > 0, \tag{5.1}$$

and either $\limsup \operatorname{diam} (f_n(U)) < 2$ *or* $\liminf \operatorname{diam} (f_n^{-1}(U)) > 2$, *then $F_n(w)$ converges locally uniformly in U to a constant function.*

The choice $w_n := \infty$ in Theorem 5.2 implies Theorem 5.1.

A similar idea works in a more general setting, where f_n maps an open set $V \subseteq \hat{\mathbb{C}}$ into itself. V does not even have to be connected. The contraction obtained by the condition $\limsup \operatorname{diam}(f_n(U)) < 2$ (or $\liminf \operatorname{diam}(f_n^{-1}(U)) > 2$) is now replaced by $f_n \in \mathscr{M}_\varepsilon(V)$ for some $\varepsilon > 0$, where

$$\mathscr{M}_\varepsilon(V) := \left\{ f(w) := \frac{aw + b}{cw + d} \colon ad - bc \neq 0, f(V) \subseteq V \backslash B(c_f, \varepsilon) \text{ for a } c_f \in \bar{V} \right\},$$

and $B(a, \rho) := \{w \in \hat{\mathbb{C}} \colon d(w, a) \leqslant \rho\}$, where d is the chordal metric on the Riemann sphere $\hat{\mathbb{C}}$. The center c_f of the omitted disk $B(c_f, \varepsilon)$ may depend on f, and it may belong to ∂V. Still $\mathscr{M}_\varepsilon(V)$ may be empty, unless V and $\varepsilon > 0$ are chosen properly.

Theorem 5.3 (Lorentzen [31]). *Let $V \subseteq \hat{\mathbb{C}}$ be open, $\varepsilon > 0$ and $\{f_n\}$ be a sequence from $\mathscr{M}_\varepsilon(V)$. If there exist two sequences $\{u_n\}$ and $\{v_n\}$ from $\hat{\mathbb{C}}$, bounded away from ∂V, such that no subsequence $\{n_k\}$ of the natural numbers gives*

$$\lim_{k\to\infty} \tau_{2k-1}(u_{n_{2k-1}}) = \lim_{k\to\infty} \tau_{2k}^{-1}(v_{n_{2k-1}}) \quad \text{and} \quad \lim_{k\to\infty} \tau_{2k}(u_{n_{2k}}) = \lim_{k\to\infty} \tau_{2k+1}^{-1}(v_{n_{2k}}),$$

for a sequence $\tau_k := f_{n_{k-1}+1} \circ f_{n_{k-1}+2} \circ \cdots \circ f_{n_k}$, then $\{F_n(w)\}$ converges locally uniformly in V to a constant function.

The limit-point case has several advantages. It is therefore of interest to identify situations where this occurs.

Theorem 5.4 (Lorentzen [30]). *Let $\{f_n\}$ be a sequence of linear fractional transformations mapping the unit disk U into itself. If either*
(i) $\sum d_n = \infty$, *where* $d_n := \inf\{1 - |f_n(w)|: w \in U\}$, *or*
(ii) $\sum \tilde{d}_n = \infty$, *where* $\tilde{d}_n := \inf\{|f_n^{-1}(w)| - 1: w \in \partial U\}$, *or*

$$\text{(iii)} \lim_{n\to\infty} r_n \frac{|\zeta_n| + 1}{|\zeta_n| - 1} \prod_{j=1}^{n-1} \kappa_j = 0, \text{ where } \kappa_j := r_j \frac{|\zeta_j|^2 - 1}{(|\zeta_j - c_{j+1}| - r_{j+1})^2} \leqslant r_j \frac{|\zeta_j| + 1}{|\zeta_j| - 1},$$

r_j is the radius of $f_j(U)$ and c_j is its center, and $\zeta_j := f_j^{-1}(\infty)$, then $F_n(w)$ converges uniformly in $\bar{U}$ to a constant function.

A different approach was taken in [37].

Theorem 5.5 (de Pree and Thron [37]). *Let $f_n(w) := (a_n w + b_n)/(c_n w + d_n)$ where $a_n d_n - b_n c_n = 1$, $|a_n| \geqslant 1$ and $\sum(|a_n| - 1)$, $\sum |b_n|$ and $\sum |d_n|$ converge. Then $\{F_n(w)\}$ converges to a constant function for all $w \neq 0$.*

Gill [7–10] (see also references therein) has studied such limit periodic sequences of functions.

6. Continued fractions $K(a_n/1)$

Continued fractions of the form $K(a_n/1)$ can be regarded as compositions of linear fractional transformations of the form $s_n(w) = a_n/(1 + w)$. That is, the class where $s_n(\infty) = 0$ and $s_n^{-1}(\infty) = -1$.

Let $V \subseteq \mathbb{C}$ be a bounded set, not necessarily open or connected. The requirement $s_n(V) \subseteq V$ for at least one function s_n, ensures that $-1 \notin \bar{V}$, and thus that -1 has a positive distance to V. In 1989, I gave a talk in Luminy, France, posing the question: What extra conditions are needed to ensure convergence of $K(a_n/1)$ if it has such a bounded value set V?

Of course, one has to avoid the situation where all $a_n = a < -\frac{1}{4}$, in which case s_n is elliptic. But does one really need more? This is still an open problem. What is known is the following:

Theorem 6.1 (Lorentzen and Ruscheweyh [32]). *Let V be an open, bounded set in $\mathbb{C}$, and let $E := \{a \in \mathbb{C}: a/(1 + V) \subseteq V\}$ contain at least two points. Further, let*

$$\partial^* V := \{w \in \partial V: -1 - w \in \partial V\}, \qquad E^* := \{a \in E: a/(1 + \partial^* V) \cap \partial^* V \neq \emptyset\}.$$

Then every continued fraction $K(a_n/1)$ from E, whose elements $\{a_n\}$ has at least one limit point $\notin E^*$, converges.

This can be seen as a considerable generalization of an early result by Roach [39]. The condition that E has at least two points is sensible. It ensures that all s_n are non-elliptic. Besides, the case where E contains only one point is simple to handle separately. However, the second condition that $\{a_n\}$ should have a limit point $\notin E^*$, we believe to be superfluous. In the Simple Parabola Theorem, for instance, we have $E = P_\alpha$ and $E^* = \partial P_\alpha$. If $\{a_n\} \subseteq P_\alpha$ is bounded, then a bounded subset of V_α is a value set for $K(a_n/1)$. Still, one does not need any restriction on a_n approaching the boundary.

7. The choice of value sets

To derive new convergence criteria for continued fractions $K(a_n/1)$, the choice of a simple value set V should be made such that it works for a large class of continued fractions. Hence, one wants the value set to be large. But not too large, since that may decrease the element set E. As an example, consider the Uniform Parabola Sequence Theorem. If all $g_n = g$ close to 1, then $V_{\alpha,n}$ gets large, but $P_{\alpha,n}$ shrinks as g increases.

If the aim is to produce good truncation error bounds, one normally wants the value sets to be small to get better bounds.

However, in frequent situations one is not at liberty to choose $\{V_n\}$ freely. One has a given continued fraction or a class $\mathscr{C}$ of continued fractions, and wants to find corresponding value sets. This is not always so easy! In the following we assume that $\mathscr{C}$ is such a class containing more than one convergent continued fraction $K(a_n/b_n)$ (and possibly divergent ones as well). It is relatively simple to see that the *limit sets*

$$L_n := \left\{ f^{(n)} := \frac{a_{n+1}}{b_{n+1}} \underset{+}{} \frac{a_{n+2}}{b_{n+2}} \underset{+}{} \cdots : \text{convergent } K(a_n/b_n) \in \mathscr{C} \right\} \tag{7.1}$$

for the class $\mathscr{C}$, constitute a sequence of value sets for every continued fraction from $\mathscr{C}$ [16]. Moreover, it follows from [16] that if V is a simple value set for the class $\mathscr{C}$ of continued fractions $K(a_n/1)$, then $L_n \subseteq \bar{V}$ for all n. More properties of limit sets can be found in [29].

Rye and Waadeland [40] studied the case where $\mathscr{C}$ consists of all continued fractions $K(a_n/1)$ from a set $E \subseteq \mathbb{C}$. As they pointed out, the natural strategy, to pick out N points from E, $E(N)$, and compute all the points in the set

$$V(N, n) := \mathop{\mathrm{K}}_{k=1}^{n} \frac{E(N)}{1} \tag{7.2}$$

involves far too much computation, even for moderate values of N and n. They therefore suggested two alternative methods to obtain numerical approximations to the limit set L (see also [13]):

Idea 1: Regard the elements a_n as generalizations of a stochastic variable. In other words, pick continued fractions at random, by picking elements a_n from E at random, and plot the approximants.

Idea 2: Use the sparse set of 2-periodic continued fractions from E, whose elements are picked from some extreme subset E^* of E.

A third idea was presented in [55], namely to use linear approximations to the values of these 2-periodic continued fractions from E^*. For all these suggestions, even if one is not able to find explicit value sets for a given class $\mathscr{C}$ of continued fractions, these pictures can help to prove that the limit sets omit certain parts of the complex plane. The Stieltjes–Vitali Theorem can then help to prove the convergence.

If $E(N)$ or E^* are finite sets with few elements, say two, then the set (7.2) may approach a beautiful fractal set as $n \to \infty$. This has been studied by Waadeland [56] and Karlsson and Wallin [19].

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